Research Statement

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My research interest lies broadly in hyperbolic geometry. Currently I am interested in the interplay between geometric properties of various types of surfaces in Euclidean space and their induced complex structures. This involves in particular methods from hyperbolic geometry, Riemann surfaces, classical Euclidean geometry, and minimal surfaces.

Introduction

In classical differential geometry, a central question has been whether abstract surfaces with given geometric features can be realized as surfaces in Euclidean space. Typical early results are Hilbert's proof that surface of constant negative curvature in Euclidean space cannot be complete [5], or the theorem of Hartman-Nirenberg that a complete flat surface in Euclidean space must be a cylinder [4]. I am particularly interested in the constraints the conformal structure of a surface (in combination with other geometric features) puts on possible realizations of the surface in space. This question has been well studied in cases where the connection between surface geometry and conformal type is particularly strong, for instance in the case of minimal and constant mean curvature surfaces. I aim to address this problem for another class of surfaces, namely surfaces with cone metrics. Two fundamental results are the following.

1. (Schwarz-Christoffel; Troyanov) Every compact Riemann surface with finitely many distinguished points and chosen angles at these points possesses a conformal flat cone metric with the prescribed cone angles at the distinguished points. The cone angles need to satisfy a sum condition, and the metric is essentially unique.

2. (Aleksandrov) Any cone metric on the sphere with cone angles less than 2π has a unique realization in Euclidean space as the boundary of a convex body.

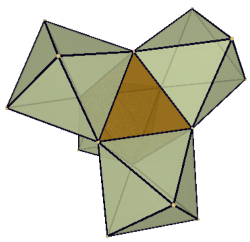
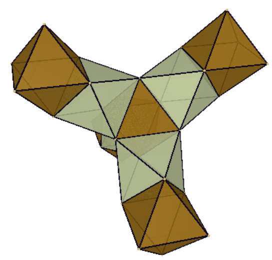
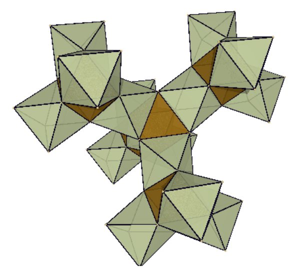
Little is known in other cases. Inspired by the rich theory of embedded triply periodic minimal surfaces, I have been investigating which cone metrics on compact Riemann surfaces of higher genus have a realization as the quotient of a triply periodic polyhedral surface in Euclidean space. Results include many examples and a classification theorem, to be discussed below. Some of the examples shed new light on existing minimal or algebraic surfaces, while others are new.

The methods used include flat structures and translation structures on Riemann surfaces, hyperbolic geometry, algebraic geometry, and the theory of cyclically branched coverings over punctured spheres.

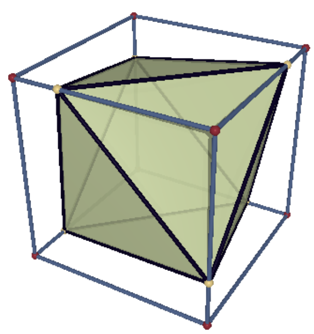
Summary of research I have done

Fermat's quartic, Octa-4, or $\{3, 8|3\}$

In [7] I introduce a triply periodic polyhedral surface, which we will call Octa-4 surface, whose quotient via its maximal group of translations is a genus $g = 3$ Riemann surface. In Schläfli symbols it is denoted as $\{3, 8|3\}$ where $\{p, q|r\}$ represents a polyhedral surface that is constructed by q regular p-gons at each vertex forming regular r-gonal holes. We say a polyhedral surface is intrinsically Platonic when it can be written by Schläfli symbols. The following figure describes its topological construction.

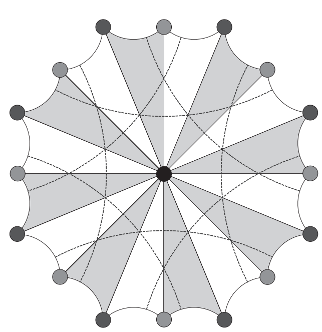
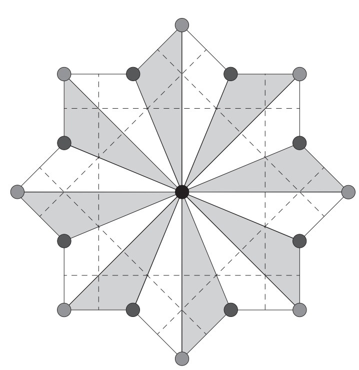




To construct the infinite polyhedral surface, to every brown octahedron we attach green octahedra on four non-adjacent faces and to each green octahedron we attach two brown octahedra on opposite faces. By placing an octahedron inside a cube where the vertex of an octahedron divides the edge of a cube by a 1:3 ratio, we prove that this surface has no self-intersection.



Theorem 1 (L)**.** *The Octa-4 surface is a geometric realization of Fermat's quartic as a triply periodic polyhedral surface.*

The dotted lines on the following figures indicate identification of edges.

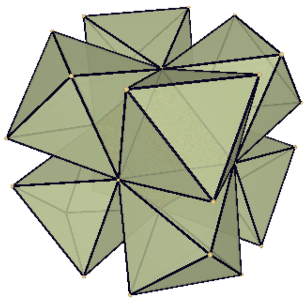
The cone metric induces hyperbolic metric where there is an order eight automorphism. This yields translational structures that are compatible with its conformal type, specifically as an eightfold cyclically branched cover over a thrice punctured sphere. Then we use translational structures to explicitly find a basis of holomorphic 1-forms with the following divisors

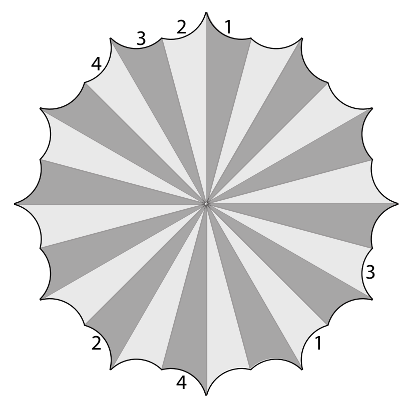
$$\begin{aligned}(\omega_1) &= & & & & 4p_3\\ (\omega_2) &= & p_1 & +p_2 & +p_2' & +p_3\\ (\omega_3) &= & 4p_1 & & & \end{aligned}$$

with which we get an algebraic equation $\omega_1^3\omega_3 - \omega_1\omega_3^3 = \omega_2^4$. Given $X^3Y - XY^3 = Z^4$, we dehomogenize the equation by setting $Y = 1$. Then $Z^4 = X^3 - X = X(X-1)(X+1)$. Via change of coordinates we map $-1, 0, 1$, and ∞ to the fourth roots of unity and get $V^4 = W^4 - 1$ and after homogenization we get $U^4 + V^4 = W^4$ which implies the Fermat's quartic.

Schoen's minimal I-WP surface, Octa-8, or $\{3, 12|3\}$

In my thesis work, I present a Platonic polyhedralization of Schoen's minimal I-WP surface which we call Octa-8. It can be written as $\{3, 12|3\}$ in Schläfli symbols and its quotient can be realized as a twelvefold cyclically branched cover over a thrice punctured sphere. I carry out a similar computation to construct a basis of holomorphic 1-forms and prove the following theorem.





$$\begin{aligned}(\omega_1) &= & & & & & & 6r\\ (\omega_2) &= & p & +q_1 & +q_2 & +q_3 & +q_4 & +r\\ (\omega_3) &= & 3p & & & & & 3r\\ (\omega_4) &= & 6p & & & & & \end{aligned}$$

Theorem 2 (L)**.** $\{3, 12|8\}$ *is conformally equivalent to Schoen's minimal I-WP surface.*

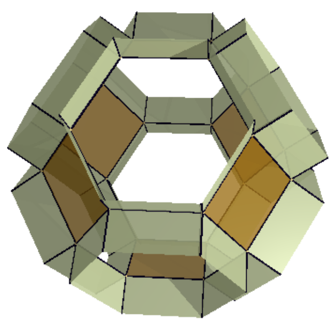
We start from the basis of 1-forms $\{\omega_i\}$ and via change of variables we have $\varphi_1 = \frac{-\omega_1+\omega_2}{2\omega_3}dh, \varphi_2 = i\frac{\omega_1+\omega_2}{2\omega_3}dh$, and $\varphi_3 = dh$. Then we get $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$ which implies that this is a minimal surface. Furthermore we define meromorphic functions whose divisors are defined as follows: $(f) := \left(\frac{\omega_1}{\omega_4}\right)$, $(g) := \left(\frac{\omega_2}{\omega_4}\right)$, and $(h) := \left(\frac{\omega_3}{\omega_4}\right)$. Then $g^3 = h^5 - h$ exhibits a threefold cover over a six-punctured sphere, that coincides with the degree three Gauss map on Schoen's minimal I-WP surface.

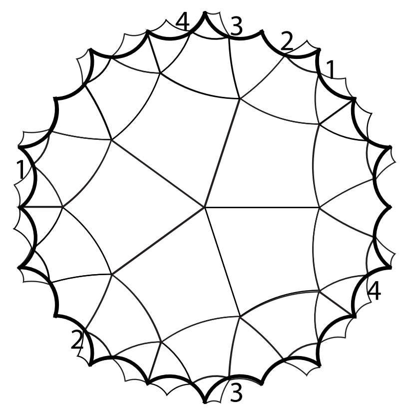
Bring's curve, $\{4,5|6\}$

Bring's curve can be written as

$$v+w+x+y+z=v^2+w^2+x^2+y^2+z^2=v^3+w^3+x^3+y^3+z^3=0$$

and has S_5 as its automorphism group, which is the largest possible automorphism group for a genus four surface. It is known to have a polyhedral description as Kepler's small stellated dodecahedron. However I provide a triply periodic polyhedral surface whose polyhedral metric induces conformal structure that is conformally equivalent to that of Bring's curve. It can be written as $\{4,5|6\}$ and its hyperbolic structure suggests that it is a fivefold cyclically branched cover over a four-punctured sphere, and the branching orders correspond to its description in [12].





Theorem 3 (L)**.** $\{4,5|6\}$ *is a triply periodic polyhedral realization of Kepler's small stellated dodecahedron, whose cone metric induces the same conformal structure.*

Classification

All three surfaces arise as the boundary surface of a polyhedron built by regular solids including Platonic, Archimedean solids, prisms, anti-prisms. For example the Octa-4 and Octa-8 are boundaries of polyhedra built by octahedra, and Kepler's small stellated dodecahedron is the boundary of a polyhedron built by truncated octahedra and hexagonal prisms. The decoration is an attempt to make Schoen's heuristic concept of a dual graph rigorous [10]. In my thesis, I show a classification of such surfaces for genus three and four. I define regular embedded graphs in Euclidean space, then propose a definition of a decoration of a graph.

We define a graph to be a one-dimensional simplicial complex in three-dimensional Euclidean space, where V is a set of vertices and E is a set of edges. We say that a graph Γ is *regular* if the number of edges incident to a vertex v is constant for all $v \in V$, if all edges have the same length, and if no two vertices share more than one edge. Given a graph, we replace the vertices and edges with three-dimensional polyhedra and glue them along their polygonal faces so that the gluing of solids embeds in $\mathbb{R}^3$. We call this a *decoration* of a graph where the boundary surface is isotopic to the δ-neighborhood of the graph. We say it is a *regular Archimedean decoration* if the solids include only Platonic solids, Archimedean solids, prisms and anti-prisms over regular p-gons and if the polyhedral surface can be written by Schläfli symbols.

Given a periodic graph, we define the genus of the graph as the genus of the quotient of its δ-neighborhood. We show that under certain regularity conditions there are only finitely many regular embedded graphs of genus g. We also show that there are finitely many regular Archimedean decorations of embedded graphs for each g.

Theorem 4 (L)**.** *There is a finite, explicit list of regular Archimedean decorations of embedded graphs of lower genus.*

Related questions

1. Several examples of cyclically branched covers over punctured spheres have given interesting geometric realizations as quotients of triply periodic surfaces in euclidean space. I would also like to find more examples and expand this to branched covers over tori.

2. I would also like to expand and address the previous question in hyperbolic space. There is a polyhedral surface whose cone metric implies that its hyperbolic structure describes the Klein's quartic as a sevenfold branched cover over a thrice punctured sphere. However this cannot be a quotient of a periodic surface in euclidean space. One can almost get Klein's quartic in hyperbolic space, but it is not clear whether one can resolve this issue.

3. Lastly I would like to strengthen the classification theorem for regular polyhedral surfaces and expand the discussion to cases of genus five.

References

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